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How composite bosons really interact

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Abstract. The aim of this paper is to clarify the conceptual difference which exists between the interactions of composite bosons and the interactions of elementary bosons. A special focus is made on the physical processes which are missed when composite bosons are replaced by elementary bosons. Although what is here said directly applies to excitons, it is also valid for composite bosons in other fields than semiconductor physics. We, in particular, explain how the two elementary scatterings – Coulomb and Pauli – of our many-body theory for composite excitons, can be extended to a pair of fermions which is not an Hamiltonian eigenstate – as for example a pair of trapped electrons, of current interest in quantum information.

PACS. 71.35.-y Excitons and related phenomena

In the 50's, theories have been developed to treat many-body effects between quantum elementary particles, fermions or bosons, and their representation in terms of Feynman diagrams has been quite enlightening to grasp the physics involved in the various terms. This many-body physics is now well explained in various textbooks [1–4].

While these theories have allowed a keen understanding of the microscopic physics of electron systems, a fundamental problem remains up to now in the case of bosons because essentially all particles called bosons are composite particles made of an even number of fermions. Various attempts have been made to get rid of the underlying fermionic nature of these bosons, through procedures known as "bosonizations" [5]. By various means, their main goal is to find a convincing way to trust the final replacement of a pair of fermions — for the simplest of these bosons — by an elementary boson, their fermionic nature being hidden in "effective scatterings", which supposedly take care of possible exchanges between the fermions from which these composite bosons are made.

A few years ago [6–8], we have decided to tackle the problem of interacting composite bosons, with as a main goal, to find a way to treat their interactions without replacing them by elementary bosons, at any stage. It is clear that a many-body theory for composite bosons is expected to be more complex than the one for elementary bosons. However, the new diagrammatic representation we have recently constructed [9], greatly helps to understand the processes involved in the various terms, by making transparent the physics they contain.

The main difficulty with interacting composite particles is the concept of interaction itself. A first — rather simple — problem is linked to the fact that, fermions being indistinguishable, there is no way to know with which fermions these composite particles are made. As a direct consequence, there is no way to identify the elementary interactions between fermions which have to be assigned to interactions *between* composite bosons: Indeed, if we consider two excitons made of two electrons $(e,\, e')$ and two holes $(h,\, h')$, there are six elementary Coulomb interactions between them: $V_{ee'}, V_{hh'}, V_{eh}, V_{e'h'}, V_{eh'}$ and $V_{e'h}$. While $(V_{ee'} + V_{hh'})$ is unambiguously a part of the interaction between the two excitons, $(V_{eh'} + V_{e'h})$ is the other part if we see the excitons as made of (e, h) and (e', h'), while this other part is $(V_{eh} + V_{e'h'})$ if we see them as made of (e, h') and (e', h). This ambiguity means that there is no clean way to transform the interacting part of an Hamiltonian written in terms of fermions, into an interaction between composite bosons. From a technical point of view, this is dramatic, because, with the Hamiltonian not written as $H_0 + V$, all our background on interacting systems, which basically relies on perturbation theory at finite or infinite order, has to be given up, so that new procedures [10] have to be constructed from scratch, to calculate the physical quantities at hand.

A second problem with composite bosons made of fermions, far more vicious than the first one, is linked to Pauli exclusion between the fermion components. While Coulomb interaction, originally a 2×2 interaction, produces many-body effects through correlation, Pauli exclusion produces this "N-body correlation" at once, even in the absence of any Coulomb process. In the case of many-body effects between elementary fermions, this Pauli "interaction" is hidden in the commutation rules for fermion operators, so that we do not see it. It is however known

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to be crucial: Indeed, for a set of electrons, it is far more important than Coulomb interaction, because it is responsible for the electron kinetic energy which dominates Coulomb energy in the dense limit. When composite bosons are replaced by elementary bosons, the effect of Pauli exclusion is supposedly taken into account by introducing a phenomenological "filling factor" which depends on density. In our many-body theory for composite bosons, this Pauli exclusion appears in a microscopical way through a dimensionless exchange scattering from which can be constructed all possible exchanges between the N composite bosons.

Since our many-body theory for composite bosons is rather new and not well known yet, many people still thinking in terms of bosonized particles with dressed interactions, it appears to us as useful to come back to the concept of interaction for composite bosons, because it is at the origin of essentially all the difficulties encountered with their many-body effects, when one thinks in a conventional way, i.e., in terms of elementary particles.

The goal of this paper is (i) to carefully study the interactions between two and three composite bosons, in order *to clarify the set of physical processes which are missed by any bosonization procedure*, whatever the choice made for the effective scatterings is, (ii) *to show how the two conceptually different scatterings* of our new many-body theory for composite excitons, namely Coulomb and Pauli, *can be extended to other types of composite bosons*, in particular the ones which are not Hamiltonian eigenstates.

This paper is organized as follows:

In a first section, we briefly recall how elementary particles interact. We also recall a few simple ideas on their many-body physics.

In a second section, we consider composite bosons made of two different fermions. We will call them "electron" and "hole", having in mind, as a particular example, the case of semiconductor excitons. We physically analyse what can be called "interactions" between two and between three of these composite bosons. We then show how these physically relevant "interactions" can be associated to precise mathematical quantities constructed from the microscopic Hamiltonian written in terms of fermions.

In a third section, we discuss, on general grounds, the limits of what can be done when composite bosons are replaced by elementary bosons [11,12], in order to identify which kind of processes are systematically missed.

In a last section, we show a natural extension of the ideas of our many-body theory for composite excitons to composite bosons which are not exact eigenstates of the Hamiltonian, for example a pair of trapped electrons, of current interest in quantum information [13,14].

This paper is definitely not a precise application of our new approach to any specific physical problem. In various publications [10,15,16], we have already shown that our exact approach produces terms which are missed when composite excitons are replaced by elementary bosons with dressed interactions, these terms being all linked to a weak treatment of carrier exchanges. Since our approach now provides a clean and secure way to tackle problems dealing with composite boson many-body effects, it appears to us as useful to clarify the conceptual breakthrough our theory provides in problems of high current interest, like the Bose-Einstein condensation of excitons [17,18] and the semiconductor optical nonlinearities in semiconductors — since photons interact with a semiconductor through the virtual excitons to which they are coupled.

1 Interaction between elementary bosons

Let us call $|\bar{i}\rangle = \bar{B}_i^\dagger|v\rangle$ a one-elementary-boson state, its creation operator $\bar{B}_i^\dagger$ being such that

$$[\bar{B}_m, \bar{B}_i^\dagger] = \delta_{m,i}. \tag{1.1}$$

The concept of interaction between these elementary bosons is associated to the idea that, if *two* of them, initially in states i and j, enter a "black box", they have some chance to get out in different states m and n (see Fig. 1a). In the "black box", one or more interactions can take place (see Figs. 1b, 1c). Moreover, since the bosons are indistinguishable, there is no way to know if the boson i becomes m or n, so that the elementary process (1b) has to be the sum of the two processes shown in Figure 1d.

From a mathematical point of view, the interaction between elementary bosons appears through a potential in their Hamiltonian, which reads

$$\bar{V} = \frac{1}{2}\sum_{mnij} \bar{\xi}^{\text{eff}}_{mnij}\, \bar{B}_m^\dagger \bar{B}_n^\dagger \bar{B}_i \bar{B}_j\ , \tag{1.2}$$

with

$$\bar{\xi}^{\text{eff}}_{mnij} = \bar{\xi}^{\text{eff}}_{nmij}\ , \tag{1.3}$$

due to the boson undistinguishability and

$$\bar{\xi}^{\text{eff}}_{mnij} = \left(\bar{\xi}^{\text{eff}}_{ijmn}\right)^*\ , \tag{1.4}$$

due to the necessary hermiticity of the Hamiltonian.

To make a link between what will be said in the following on composite bosons, it is of interest to note that, if the system Hamiltonian $\bar{H}$ reads $\bar{H} = \bar{H}_0 + \bar{V}$, with $\bar{H}_0 = \sum_i \bar{E}_i\, \bar{B}_i^\dagger \bar{B}_i$ and $\bar{V}$ given by equation (1.2), we have

$$[\bar{H}, \bar{B}_i^\dagger] = \bar{E}_i\, \bar{B}_i^\dagger + \bar{V}_i^\dagger\ , \tag{1.5}$$

with $\bar{V}_i^\dagger|v\rangle = 0$, while

$$[\bar{V}_i^\dagger, \bar{B}_j^\dagger] = \sum_{mn} \bar{\xi}^{\text{eff}}_{mnij}\, \bar{B}_m^\dagger \bar{B}_n^\dagger\ . \tag{1.6}$$

This leads to an Hamiltonian matrix element in the two-boson subspace given by

$$\langle v|\bar{B}_m\bar{B}_n H \bar{B}_i^\dagger \bar{B}_j^\dagger|v\rangle = 2[(E_i + E_j)\,\delta_{mnij} + \bar{\xi}^{\text{eff}}_{mnij}]\ , \tag{1.7}$$

the scalar product of two-elementary-boson states being such that

$$\langle v|\bar{B}_m\bar{B}_n \bar{B}_i^\dagger \bar{B}_j^\dagger|v\rangle = 2\delta_{mnij} = \delta_{m,i}\delta_{n,j} + \delta_{m,j}\delta_{n,i}\ . \tag{1.8}$$

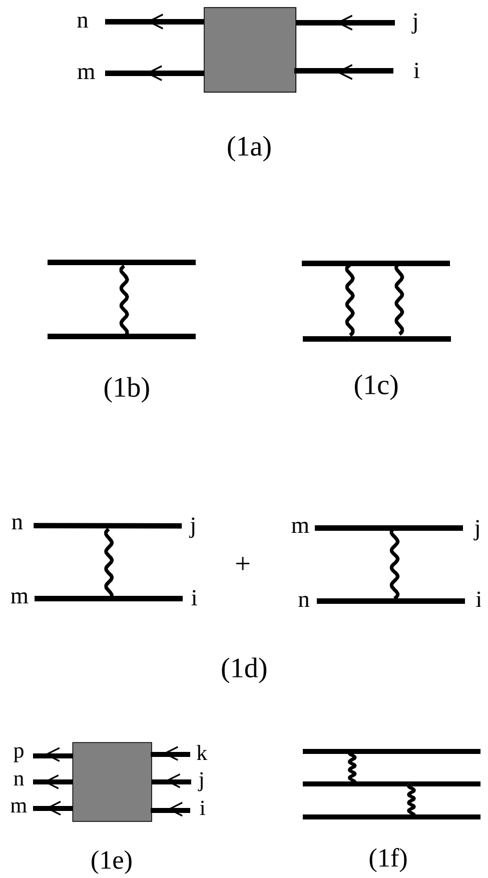

Fig. 1. (a) (resp. (e)): basic diagrams for the interactions of two (resp. three) *elementary* bosons. Between two composite bosons, one, two, or more interactions can exist as in (b) and (c), while two interactions at least are necessary (see (f)) to find three composite bosons in "out" states (m, n, p) different from the "in" states (i, j, k). Due to the boson undistinguishability, the elementary scattering between two bosons must be invariant under a $(m \leftrightarrow n)$ and/or a $(i \leftrightarrow j)$ permutation, as shown in (d).

If we now have three bosons entering the "black box", two interactions at least are necessary, in order to find these bosons out of the box, all three in a state different from the initial one (see Figs. 1e, 1f). Since $\bar{\xi}^{\text{eff}}_{mnij}$ has the dimension of an energy, the second scattering of this two-interaction process has to appear along with an energy denominator.

2 Interactions between composite bosons

We now consider a composite boson made of two different fermions. Let us call them "electron" and "hole". The case of composite bosons made of a pair of identical fermions will be considered in the last part of this work. We label the possible states of this composite boson by i.

2.1 Two composite bosons

We start by considering two composite bosons in states i and j. From a conceptual point of view, an "interaction" is a physical process which allows to bring these bosons into two different states, m and n. What can possibly happen in the "black box" of Figure 2a, to produce such a state change?

2.1.1 Pure carrier exchange

The simplest process is, for sure, just a carrier exchange, either with the holes as in Figure 2b, or with the electrons as in Figure 2c. Since the two are physically similar, we expect them to appear equally in a scattering λ_{mnij} based on this pure exchange (see Fig. 2d). It is of interest to note that the electron exchange of Figure 2c is equivalent to a hole exchange, with the (m, n) states permuted (see Fig. 2c').

If this carrier exchange is repeated, we see from Figure 2e that two hole exchanges reduce to an identity, i.e., no scattering at all, while a hole exchange followed by an electron exchange results in a (m, n) permutation, i.e., again no scattering at all for indistinguishable particles (see Fig. 2f).

Let us now show how we can make appearing the λ_{mnij} exchange scattering formally. In view of Figure 2d, this scattering has to read

$$2\lambda_{mnij} = \lambda \begin{pmatrix} n & j \\ m & i \end{pmatrix} + \lambda \begin{pmatrix} m & j \\ n & i \end{pmatrix} \ , \qquad (2.1)$$

where $\lambda \begin{pmatrix} n & j \\ m & i \end{pmatrix}$ corresponds to the hole exchange of Figure 2b, the excitons m and i having the same electron,

$$\lambda \begin{pmatrix} n & j \\ m & i \end{pmatrix} = \int d\mathbf{r}_e\, d\mathbf{r}_h\, d\mathbf{r}_{e'}\, d\mathbf{r}_{h'}\, \langle n|\mathbf{r}_{e'}\mathbf{r}_h\rangle\langle m|\mathbf{r}_e\mathbf{r}_{h'}\rangle\langle \mathbf{r}_e\mathbf{r}_h|i\rangle\langle \mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle \ , \qquad (2.2)$$

where $\langle \mathbf{r}_e\mathbf{r}_h|i\rangle$ is the wave function of the one-boson state $|i\rangle$. Note that the prefactor 2 of equation (2.1), which could be included in the definition of the Pauli scattering, is physically linked to the fact that *two* exchanges are possible in an electron-hole pair, namely a hole exchange and an electron exchange. It is of interest to note that, in the case of one electron and one exciton, as in problems dealing with trions, these Pauli scatterings appear without any prefactor 2 because the exciton can only exchange its electron with the electron gas.

If these one-boson states are orthogonal, $\langle m|i\rangle = \delta_{m,i}$, it is tempting to introduce the deviation-from-boson operator D_{mi} defined as

$$D_{mi} = \delta_{m,i} - [B_m, B_i^\dagger], \qquad (2.3)$$

where $B_i^\dagger$ is the creation operator for the one-boson state $|i\rangle = B_i^\dagger|v\rangle$. For $\delta_{m,i} = \langle m|i\rangle$, this operator is such that

$$D_{mi}|v\rangle = 0, \qquad (2.4)$$

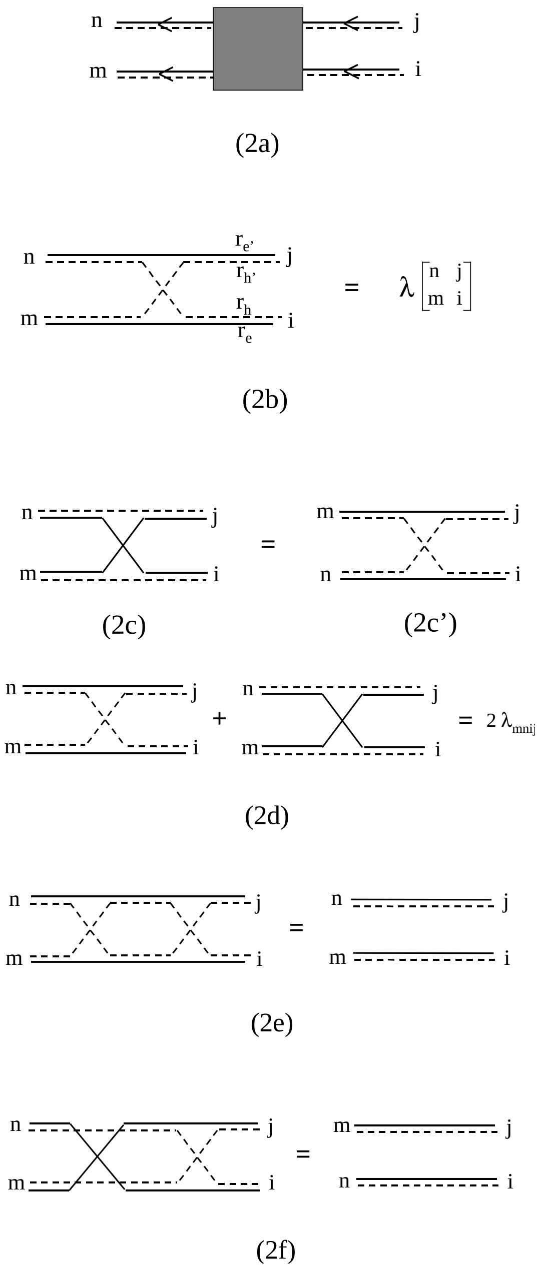

Fig. 2. (a) Basic diagram for the interaction of two *composite* bosons made of an electron (solid line) and a hole (dashed line). (b) Elementary *hole* exchange $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ between the "in" composite bosons (i, j) and the "out" composite bosons (m, n). (c) Elementary *electron* exchange between the same composite bosons as the ones of (b). As shown in (c'), this electron exchange is equivalent to a hole exchange with (m, n) changed into (n, m). (d) Due to the undistinguishability of the fermions forming the composite bosons, the elementary Pauli scattering λ_{mnij} between two composite bosons must be invariant under a $(m \leftrightarrow n)$ and/or $(i \leftrightarrow j)$ permutation. Due to (c, c'), this Pauli scattering must include a hole exchange and an electron exchange. (e) Two hole exchanges reduce to an identity. (f) One hole exchange followed by an electron exchange reduces to a (m, n) permutation: indeed, the resulting composite boson m is made with the same fermions as j. Note that all these exchange processes are missed when composite bosons are replaced by elementary bosons.

while its commutator with another boson creation operator makes appearing the exchange or Pauli scatterings we want, through

$$[D_{mi}, B_j^\dagger] = 2\sum_n \lambda_{mnij}\, B_n^\dagger \ , \tag{2.5}$$

as easy to see by calculating the scalar product of the two-boson states $\langle v|B_m B_n B_i^\dagger B_j^\dagger|v\rangle$, using either the set of commmutators (2.3, 5) or the two-composite-boson wave function,

$$\langle \mathbf{r}_{e'}\mathbf{r}_{h'}, \mathbf{r}_e\mathbf{r}_h|B_i^\dagger B_j^\dagger|v\rangle = \frac{1}{2}\left[\langle \mathbf{r}_e\mathbf{r}_h|i\rangle\langle \mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle - \langle \mathbf{r}_{e'}\mathbf{r}_h|i\rangle\langle \mathbf{r}_e\mathbf{r}_{h'}|j\rangle \ + (i \leftrightarrow j)\right]. \tag{2.6}$$

This wave function is indeed invariant by $(i \leftrightarrow j)$, as imposed by $B_i^\dagger B_j^\dagger = B_j^\dagger B_i^\dagger$ for $B^\dagger$'s being products of fermion operators. It also changes sign under a $(\mathbf{r}_e, \mathbf{r}_{e'})$ exchange, as required by Pauli exclusion.

This leads to

$$\langle v|B_m B_n B_i^\dagger B_j^\dagger|v\rangle = 2[\delta_{mnij} - \lambda_{mnij}]. \tag{2.7}$$

This equation actually shows that the two-composite-boson states are nonorthogonal. This is just a bare consequence of the fact that these composite-boson states form an overcomplete basis [19]: Indeed, the composite-boson creation operators $B_i^\dagger$ are such that

$$B_i^\dagger B_j^\dagger = -\sum_{mn} \lambda_{mnij}\, B_m^\dagger B_n^\dagger \ , \tag{2.8}$$

easy to show by combining the fermion pairs in a different way.

Due to $B_i^\dagger B_j^\dagger = B_j^\dagger B_i^\dagger$, equation (2.7) also shows that

$$\lambda_{mnij} = \lambda_{mnji} = \lambda_{ijmn}^* \ . \tag{2.9}$$

Finally, from the closure relation for one-boson states, $\sum_i |i\rangle\langle i| = I$, it is easy to check that two exchanges reduce to an identity, i.e.,

$$\sum_{rs} \lambda_{mnrs}\lambda_{rsij} = \delta_{mnij} \ , \tag{2.10}$$

with δ_{mnij} given in equation (1.8), as physically expected from Figures 2e, 2f.

2.1.2 Direct and exchange Coulomb scatterings

If the two fermions are charged particles, another way for these two composite bosons to interact is via Coulomb interaction between their carriers. The simplest of these interactions is a set of direct processes in which the out excitons (m, n) are made with the same pair as the "in" excitons (i, j) (see Figs. 3a, 3b). However, here again, as the carriers are indistinguishable, these processes must appear

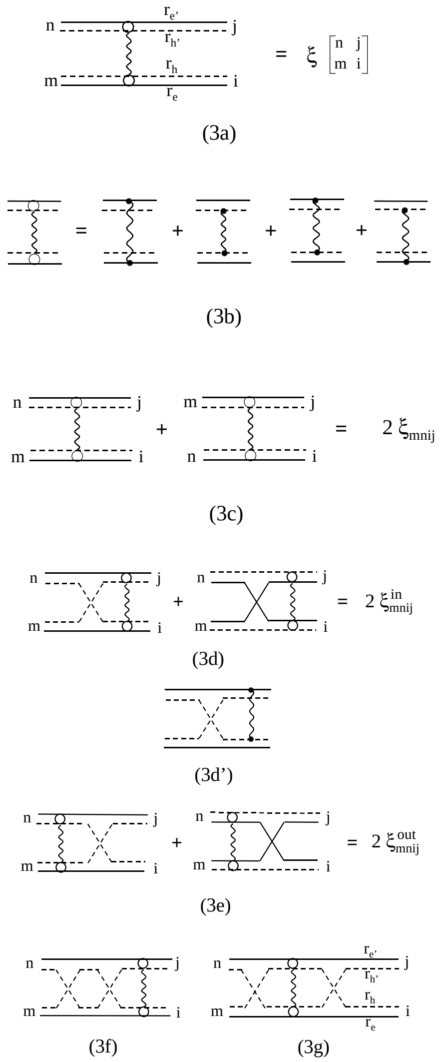

Fig. 3. (a) Elementary *direct* Coulomb scattering $\xi\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ between two composite bosons. (b) In this direct Coulomb scattering, enter the e–e, h–h as well as two e–h Coulomb interactions. (c) Due to the undistinguishability of the fermions forming the composite bosons, the direct Coulomb scattering ξ_{mnij} between two composite bosons must be invariant under a ($m \leftrightarrow n$) and/or ($i \leftrightarrow j$) permutation, so that it is composed of two elementary direct Coulomb scatterings. (d) The "in" Coulomb scattering ξ^{in}_{mnij} corresponds to a direct Coulomb scattering followed by a carrier exchange. As shown in (d'), the electron-hole Coulomb interaction of ξ^{in}_{mnij} is *between* the "in" composite bosons, but inside the "out" ones. (e) The "out" Coulomb scattering ξ^{out}_{mnij} corresponds to a carrier exchange followed by a direct Coulomb interaction. (f) Processes in which the direct Coulomb interaction is followed by two hole exchanges reduce to a direct process. (g) Processes in which the hole exchanges are on both sides of the Coulomb direct interaction are physically strange because their electron-hole parts are "inside" both, the "in" and the "out" composite bosons, so that they are already counted in these composite bosons: We never find these strange processes appearing in physical quantities resulting from many-body effects between composite bosons.

in a scattering in which m and n are not differentiated, as in Figure 3c.

In view of Figures 3a, 3c, this direct Coulomb scattering must read

$$2\xi_{mnij} = \xi\begin{pmatrix} n & j \\ m & i \end{pmatrix} + \xi\begin{pmatrix} m & j \\ n & i \end{pmatrix} , \qquad (2.11)$$

where, due to Figure 3a, $\xi\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ is given by

$$\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix} = \int d\mathbf{r}_e\, d\mathbf{r}_h\, d\mathbf{r}_{e'}\, d\mathbf{r}_{h'}\, \langle n|\mathbf{r}_{e'}\mathbf{r}_{h'}\rangle\langle m|\mathbf{r}_e\mathbf{r}_h\rangle \times V(\mathbf{r}_e\mathbf{r}_h;\mathbf{r}_{e'}\mathbf{r}_{h'})\,\langle\mathbf{r}_e\mathbf{r}_h|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle ,$$

$$V(\mathbf{r}_e\mathbf{r}_h;\mathbf{r}_{e'}\mathbf{r}_{h'}) = V_{ee}(\mathbf{r}_e,\mathbf{r}_{e'}) + V_{hh}(\mathbf{r}_h,\mathbf{r}_{h'}) + V_{eh}(\mathbf{r}_e,\mathbf{r}_{h'}) + V_{eh}(\mathbf{r}_{e'},\mathbf{r}_h) . \qquad (2.12)$$

The potential $V(\mathbf{r}_e\mathbf{r}_h;\mathbf{r}_{e'}\mathbf{r}_{h'})$ is just the sum of the Coulomb interactions between an electron-hole pair made of (e,h) and an electron-hole pair made of (e',h'). Note that, this Coulomb scattering being *direct*, the interactions are between both, the "in" composite bosons (i,j) and the "out" composite bosons (m,n). From equations (2.11, 12), we see that this direct Coulomb scattering is such that

$$\xi_{mnij} = \xi_{nmij} = (\xi_{ijmn})^* . \qquad (2.13)$$

Let us now make appearing this direct Coulomb scattering ξ_{mnij} in a formal way. If the one-boson states $|i\rangle$ are eigenstates of the Hamiltonian, i.e., if

$$(H - E_i)\, B_i^\dagger |v\rangle = 0 , \qquad (2.14)$$

it is tempting to introduce the "creation potential" $V_i^\dagger$ defined as

$$V_i^\dagger = [H, B_i^\dagger] - E_i B_i^\dagger . \qquad (2.15)$$

Due to equation (2.14), this operator is such that

$$V_i^\dagger |v\rangle = 0 . \qquad (2.16)$$

If, as for the Pauli scattering λ_{mnij}, we consider the commutator of this "creation potential" with another boson creation operator, we make appearing the direct Coulomb scatterings we want, through

$$[V_i^\dagger, B_j^\dagger] = \sum_{mn} \xi_{mnij}\, B_m^\dagger B_n^\dagger . \qquad (2.17)$$

The derivation of this result, without taking an explicit form of the Hamiltonian, is however not as easy as the one for λ_{mnij}, namely equation (2.5), because, due to the overcompleteness of the composite-boson states which follows from equation (2.8), the ξ_{mnij} scattering of equation (2.17) can as well be replaced by $(-\xi^{\text{in}}_{mnij})$, where ξ^{in}_{mnij} is an exchange Coulomb scattering defined as, (see Fig. 3d),

$$\xi^{\text{in}}_{mnij} = \sum_{rs} \lambda_{mnrs}\, \xi_{rsij} . \qquad (2.18)$$

Consequently, this direct scattering ξ_{mnij} cannot be related to a precise matrix element as simply as for λ_{mnij} in equation (2.7). Indeed, if we consider the matrix element of the Hamiltonian H between two-composite-boson states, we find, depending if H acts on the right or on the left,

$$\begin{aligned}\langle v|B_m B_n H B_i^\dagger B_j^\dagger|v\rangle &= 2[(E_i+E_j)(\delta_{mnij}-\lambda_{mnij}) \\ &\quad +(\xi_{mnij}-\xi^{\rm in}_{mnij})] \\ &= 2[(E_m+E_n)(\delta_{mnij}-\lambda_{mnij}) \\ &\quad +(\xi_{mnij}-\xi^{\rm out}_{mnij})]\ , \end{aligned} \tag{2.19}$$

where $\xi^{\rm out}_{mnij}$ is also an exchange Coulomb scattering, this time defined as, (see Fig. 3e),

$$\xi^{\rm out}_{mnij} = \sum_{rs} \xi_{mnrs}\,\lambda_{rsij}\ . \tag{2.20}$$

Due to equation (2.19), these two exchange Coulomb scatterings, $\xi^{\rm in}$ and $\xi^{\rm out}$, are linked by

$$\xi^{\rm in}_{mnij} - \xi^{\rm out}_{mnij} = (E_m+E_n-E_i-E_j)\lambda_{mnij}\ , \tag{2.21}$$

while, due to equations (2.9, 13), they are such that

$$\xi^{\rm in}_{mnij} = \xi^{\rm in}_{nmij} = \left(\xi^{\rm out}_{ijmn}\right)^*\ . \tag{2.22}$$

From the definitions of ξ_{mnij} and λ_{mnij} and the closure relation for one-boson states, the "in" exchange scattering $\xi^{\rm in}_{mnij}$, shown in Figure 3d, in fact reads as ξ_{mnij} with $\langle n|\mathbf{r}_{e'}\mathbf{r}_{h'}\rangle\langle m|\mathbf{r}_e\mathbf{r}_h\rangle$ replaced by $\langle n|\mathbf{r}_{e'}\mathbf{r}_h\rangle\langle m|\mathbf{r}_e\mathbf{r}_{h'}\rangle$. We see that $\xi^{\rm in}_{mnij}$ contains electron-hole Coulomb interactions which are *between* the "in" states (i,j), but no more *between* the "out" states (m,n) (see Fig. 3d').

In the same way, the "out" exchange scattering $\xi^{\rm out}_{mnij}$, shown in Figure 3e, reads as ξ_{mnij} with $\langle\mathbf{r}_e\mathbf{r}_h|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle$ replaced by $\langle\mathbf{r}_e\mathbf{r}_{h'}|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_h|j\rangle$; so that its electron-hole Coulomb interactions are between the "out" states (m,n) but no more between the "in" states (i,j).

$\xi^{\rm in}_{mnij}$ and $\xi^{\rm out}_{mnij}$ are Coulomb scatterings with one exchange. If we now consider two exchanges, we can think of them either on the same side as in Figure 3f or on both sides as in Figure 3g. Two exchanges reducing to an identity, if these two exchanges are on the same side, it is just the same as no exchange at all. On the opposite, if they are on both sides, we end with something very strange from a physical point of view. Indeed, the scattering shown in Figure 3g reads

$$\begin{aligned}\int d\mathbf{r}_e\, d\mathbf{r}_h\, d\mathbf{r}_{e'}\, d\mathbf{r}_{h'}\, &\langle n|\mathbf{r}_{e'}\mathbf{r}_{h'}\rangle\langle m|\mathbf{r}_e\mathbf{r}_h\rangle \\ \times\ [V_{ee}(\mathbf{r}_e,\mathbf{r}_{e'}) &+ V_{hh}(\mathbf{r}_h,\mathbf{r}_{h'}) + V_{eh}(\mathbf{r}_e,\mathbf{r}_h) \\ &+ V_{eh}(\mathbf{r}_{e'},\mathbf{r}_{h'})]\,\langle\mathbf{r}_e\mathbf{r}_h|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle\ . \end{aligned} \tag{2.23}$$

So that the electron-hole interactions V_{eh} are not between the composite bosons of any side. Being "inside" both composite bosons, these V_{eh} interactions are already included in the composite bosons themselves. Consequently, there is no physical reason for them to appear once more in a scattering *between* these composite particles. This leads us to think that this type of exchange Coulomb scattering should not appear in the final expression of physical many-body quantities involving composite bosons. And, indeed, they do not appear in the problems we have up to now considered.

It is of importance to stress that there is only one physically reasonable Coulomb scattering *between* composite bosons, namely ξ_{mnij}, because its electron-hole parts are unambiguously interactions *between* the composite bosons on both sides. The relevant way to see the two other Coulomb scatterings, $\xi^{\rm in}_{mnij}$ and $\xi^{\rm out}_{mnij}$, is as a succession of a (direct) Coulomb scattering before or after a carrier exchange. ξ_{mnij} and λ_{mnij} actually form the two elementary scatterings, necessary to describe *any* kind of interaction between composite bosons. $\xi^{\rm in}_{mnij}$ and $\xi^{\rm out}_{mnij}$ are just two, among many other possible combinations of these two elementary scatterings. This is going to become even more transparent for the interactions between three composite bosons.

2.2 Three composite bosons

We now consider what can be called interaction in the case of three composite bosons, i.e., what physical processes can transform the composite bosons (i,j,k) into the composite bosons (m,n,p) (see Fig. 4a). If there is no common state between (i,j,k) and (m,n,p), all three composite bosons have to be "touched" in some way by this interaction, in order to change state.

2.2.1 Pure carrier exchange

As for two composite bosons, the simplest "interaction" between three composite bosons is surely a carrier exchange. A possible one is shown in Figure 4b, with some of its equivalent representations shown in Figures 4c, 4d: It is easy to check that, in these three diagrams, the composite boson p is made with the same electron as j and the same hole as k.

We can think of drawing diagram (4b) with the electron/hole lines exchanged. As shown in Figure 4e, this is however equivalent to a permutation of the boson indices: Indeed, in the two diagrams of this figure, the m boson has the same electron as j and the same hole as i.

It is also of interest to note that the "Shiva diagram" for three-body exchange shown in Figure 4b can actually be decomposed, in various ways, into carrier exchanges between two composite bosons only: Indeed, diagram (4c) can be drawn as (4f) and diagram (4d) as (4g), so that

$$\lambda\begin{pmatrix} p & k \\ n & j \\ m & i\end{pmatrix} = \sum_r \lambda\begin{pmatrix} n & k \\ p & r\end{pmatrix}\lambda\begin{pmatrix} r & j \\ m & i\end{pmatrix} = \sum_s \lambda\begin{pmatrix} n & s \\ m & i\end{pmatrix}\lambda\begin{pmatrix} p & j \\ s & k\end{pmatrix} \tag{2.24}$$

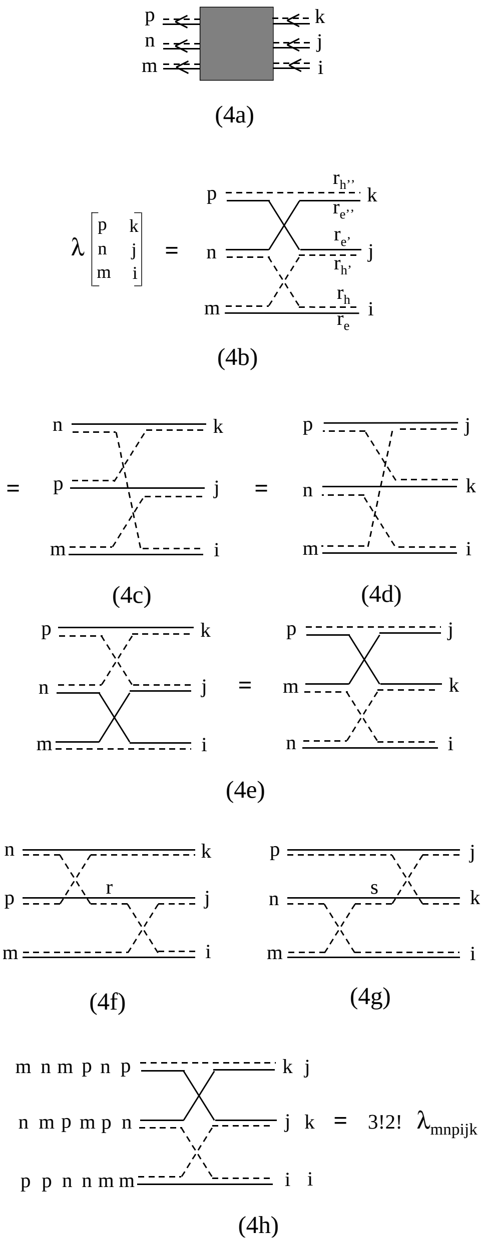

Fig. 4. (a) Basic diagram for the interaction of three composite bosons. (b) "Shiva diagram" for the carrier exchange of three composite bosons. This diagram can be redrawn as in Figures (c, d): in all these diagrams, the m composite boson has the same electron as i and the same hole as j. (e) The Shiva diagram with the electron-hole lines exchanged corresponds to a permutation of the boson indices. The Shiva diagram between three composite bosons (b) can be drawn as a succession of carrier exchanges between two composite bosons. Indeed, (c) is nothing but (f), while (d) is nothing but (g). (h) Due to the undistinguishability of the fermions forming the composite bosons, the elementary Pauli scattering λ_{mnpijk} between three composite bosons must be invariant under a (m, n, p) and/or (i, j, k) permutation. It thus contains the $6 \times 2 = 12$ processes shown on this figure. Note that, in the case of elementary bosons, two Coulomb interactions at least are necessary to have all three bosons changing state, so that the pure exchange processes shown in this figure are systematically missed when composite bosons are replaced by elementary bosons.

Since the composite bosons are made with indistinguishable particles, such a three-body exchange must however appear in a symmetrical way through a scattering λ_{mnpijk} which must read

$$3!\,2!\,\lambda_{mnpijk} = \lambda \begin{pmatrix} p & k \\ n & j \\ m & i \end{pmatrix} + 11 \text{ similar terms} , \quad (2.25)$$

obtained by permutating (m, n, p) and (i, j, k) (see fig.4h), all the other positions of (m, n, p) and (i, j, k) being topologically equivalent to one of these 3!2! terms. On that respect, it is of interest to note that the factor of 2, in the definition (2.1) of the Pauli scattering between two composite bosons λ_{mnij}, is just 2!1!. Due to Figure 4b, the elementary exchange between three composite bosons simply reads

$$\lambda \begin{pmatrix} p & k \\ n & j \\ m & i \end{pmatrix} = \int d\{d\mathbf{r}\}\, \langle p|\mathbf{r}_{e'}\mathbf{r}_{h''}\rangle\langle n|\mathbf{r}_{e''}\mathbf{r}_h\rangle$$
$$\times\, \langle m|\mathbf{r}_e\mathbf{r}_{h'}\rangle\langle\mathbf{r}_e\mathbf{r}_h|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle\langle\mathbf{r}_{e''}\mathbf{r}_{h''}|k\rangle\ . \quad (2.26)$$

This three-body Pauli scattering λ_{mnpijk} in particular appears in the scalar product of three-composite-boson states,

$$\langle v|B_m B_n B_p B_i^\dagger B_j^\dagger B_k^\dagger|v\rangle =$$
$$\delta_{mnpijk} - 2(\delta_{m,i}\lambda_{npjk} \ +8 \text{ permutations}) + 12\lambda_{mnpijk}\ , \quad (2.27)$$

with $\delta_{mnpijk} = \delta_{m,i}\delta_{n,j}\delta_{p,k}$ +5 permutations, as possible to check either directly from the explicit value of the composite boson wave function, or by using a commutator technique based on equations (2.3, 5) and on

$$3\lambda_{mnpijk} = \sum_r [\lambda_{mnri}\lambda_{prjk} + \lambda_{mnrj}\lambda_{prik} + \lambda_{mnrk}\lambda_{prij}]\ , \quad (2.28)$$

which makes use of equation (2.24).

2.2.2 One Coulomb scattering

If we now consider processes with one Coulomb scattering only, it is necessary to have one additional exchange process at least, to possibly "touch" the three composite bosons: See for example the process of Figure 5a, which precisely reads

$$\sum_s \lambda \left(\begin{smallmatrix} p & k \\ n & s \end{smallmatrix}\right) \xi \left(\begin{smallmatrix} s & j \\ m & i \end{smallmatrix}\right) = \int \{d\mathbf{r}\}\, \langle p|\mathbf{r}_{e''}\mathbf{r}_{h'}\rangle\langle n|\mathbf{r}_{e'}\mathbf{r}_{h''}\rangle\langle m|\mathbf{r}_e\mathbf{r}_h\rangle$$
$$\times\ V(\mathbf{r}_e\mathbf{r}_h;\mathbf{r}_{e'}\mathbf{r}_{h'})\,\langle\mathbf{r}_e\mathbf{r}_h|i\rangle\langle\mathbf{r}_{e'}\mathbf{r}_{h'}|j\rangle\langle\mathbf{r}_{e''}\mathbf{r}_{h''}|k\rangle. \quad (2.29)$$

Of course, we can also have one Coulomb and two exchanges, as obtained by adding one Coulomb interaction in the three-body Shiva diagram of Figure 4b (see Fig. 5b): in the process of Figure 5b, the "out" composite bosons are all constructed in a different way, while in the one of Figure 5a, one composite boson, among the three, stays made with the same fermions.

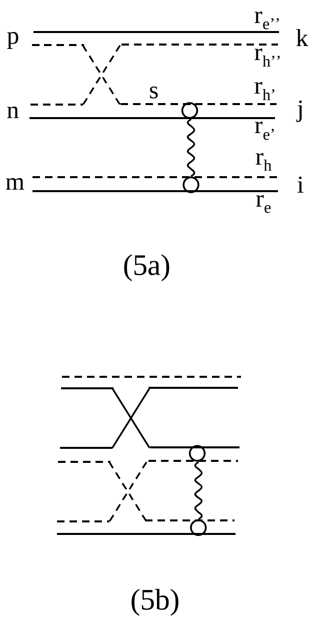

Fig. 5. Processes in which enters *one* direct Coulomb scattering. In order to have all three composite bosons changing state, these processes must also contain one (a) or two (b) carrier exchanges. Note that such processes with *one* Coulomb interaction only do not exist for elementary bosons, so that they are systematically missed when composite bosons are replaced by elementary bosons.

2.2.3 Two Coulomb scatterings

Finally, as in the case of elementary bosons, it is also possible to "touch" the three composite bosons (i,j,k) by two direct Coulomb processes, as in Figure 6a. Of course, additional fermion exchanges can take place, if the "in" and "out" bosons are made with different pairs. From a topological point of view, the processes in which the three "out" composite bosons are made with different pairs can be constructed from the Shiva diagram of Figure 4b, with the two direct Coulomb scatterings being a priori at any position, i.e., on the same side as in Figures 6b, 6c, or on both sides as in Figure 6d. On the opposite, processes in which one "out" composite boson is made with the same fermions as one of the "in" composite bosons can be constructed from the exchange diagram of Figure 2b, one of the two direct Coulomb scatterings having however to "touch" this unchanged pair, as in Figures 6e, 6f, in order to have this composite boson changing state.

2.3 Some general comments based on dimensional arguments

The qualitative analysis of what can possibly happen to two or to three composite bosons has led us to draw very many possible processes able to make them changing states. It is however of importance to note that all these complicated processes can be constructed just with two elementary scatterings, $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ and $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$, i.e., a

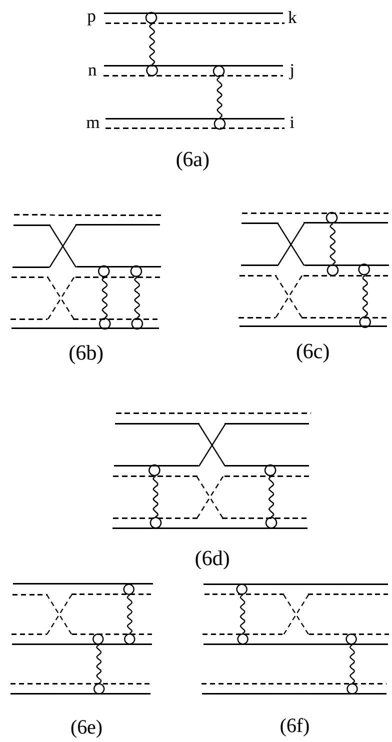

Fig. 6. Processes in which enter *two* Coulomb interactions, either through direct scatterings as in (a), or through a mixture of direct and exchange processes as in (b–f). All these processes can be written in terms of the two elementary scatterings for composite bosons, namely the direct Coulomb scattering $\xi\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ and the Pauli scattering $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$.

pure fermion exchange and a clean direct Coulomb interaction *between* two composite bosons — which is the only Coulomb process which is between the composite bosons of both sides, unambiguously.

$\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ is a scattering in the usual sense, i.e., it has the dimension of an energy. This in particular means that each time a new $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ appears in a physical quantity, a new energy denominator has also to appear; on the opposite, $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ is an unconventional "scattering" because it is dimensionless. In addition, depending on the way a new Pauli scattering appears, it can either "kill" the preceding one as in equation (2.10), or help to mix more composite bosons as in equation (2.27).

With respect to the possible goals of a many-body calculation, this makes them playing very different roles. If the relevant energies are the detunings — as in problems dealing with optical nonlinearities — the energy denominator which appears with a new $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ is a sum of detunings, so that, for unabsorbed photons, i.e., large detunings, we just have to look for processes in which enters the smallest amount of ξ's.

If we are interested in density effects, this is more subtle. The dominant terms at small density are dominated by processes in which enters the smallest amount of particles, i.e., diagrams with the smallest amount of lines. In the case of elementary bosons, we need one scattering to connect two lines, two scatterings to connect three lines, and so on . . . (see Figs. 1b, 1f), so that each new line goes with a new energy denominator. This is no more true for composite bosons: Indeed, we can connect lines in the absence of any Coulomb scattering, as in Figure 4b. Moreover, while, with exchanges alone, to connect two lines we need one Pauli scattering and to connect three lines we need two, these Pauli scatterings have to be put in very specific positions, otherwise they "destroy" themselves. Consequently, in order to generate a density expansion in a system of composite bosons, the number of ξ or λ scatterings is not a relevant quantity. Instead, we must start with the appropriate number of composite-boson lines (two for terms at lowest order in density, three for the next order terms, and so on . . .) and construct the possible connections between these lines, using $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ and/or $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$.

Of course, all this can be qualified of wishful thinking or handwaving arguments. These qualitative remarks are however of great help to identify the physics we want to describe through its visualization in a new set of "Shiva diagrams". A hard mathematical derivation of all these intuitive thinkings can always be recovered by calculating the physical quantity at hand, expressed in terms of composite boson operators, through matrix elements like $\langle v|B_{m_N}\cdots B_{m_1}\, f(H)\, B^\dagger_{i_1}\cdots B^\dagger_{i_N}|v\rangle$. To calculate such a quantity, we first push the Hamiltonian depending quantity $f(H)$ to the right, using $[f(H), B^\dagger_i]$ which can be deduced from equations (2.15, 17) for any function f. This makes appearing a set of direct Coulomb scatterings $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$. The remaining scalar products of N-composite-boson states are then calculated using equations (2.3, 5). This makes appearing a set of Pauli scatterings $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$. Note that, in this procedure, the ξ's are all together on the right, while the λ's are all together on the left (or the reverse if we push $f(H)$ to the left). This in particular avoids spurious mixtures of ξ's and λ's like the one of Figure 3g.

3 Conceptual problems with bosonization

It is of course an appealing idea to try to find a way to replace composite bosons by elementary bosons, because well known textbook techniques can then be used to treat their many-body effects. In view of Section 2, it is however clear that such a replacement raises various problems:

(i) While elementary-boson states are orthogonal, the composite boson ones are not (see Eqs. (2.7, 27)).

(ii) This is linked to the fact that, while elementary-boson states form a complete set, the set of composite-boson states is overcomplete.

(iii) Only one elementary scattering between two elementary bosons exists, namely ξ^{eff}_{mnij}. In the case of composite bosons, we have identified three scatterings having the dimension of an energy, namely ξ_{mnij}, ξ^{in}_{mnij} and ξ^{out}_{mnij}, plus one dimensionless scattering λ_{mnij}, all these scatterings being possibly constructed on a hole exchange $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ and a direct Coulomb scattering $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$. Consequently, between composite bosons, there are two fully independent scattering processes, the elementary bosons having one only.

(iv) While all the complicated processes which can exist with three composite bosons can be decomposed in terms of $\xi\begin{pmatrix} n & j \\ m & i \end{pmatrix}$ and $\lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix}$, it is necessary to introduce additional interaction potentials between three elementary bosons in the Hamiltonian, if we want to take them into account. And so on, if we are interested in processes involving four, five, . . . bosons, as necessary for higher order terms in the density.

Among all these problems, the overcompleteness of composite-boson states is for sure the major one because we are not used to work with an overcomplete basis. Let us consider it first.

3.1 Nonorthogonality and overcompleteness

These two problems are of course linked, the overcompleteness generating the nonorthogonality of the composite-boson states. However, the overcompleteness is far more difficult to handle. Just to grasp the difficulty, consider a 2D plane. To represent it, we can use the standard orthogonal basis $(\mathbf{x}, \mathbf{y})$ but we can as well use any two vectors $(\mathbf{x}', \mathbf{y}')$ which are not colinear. From them, we can either construct two orthogonal vectors, for example $(\mathbf{x}'', \mathbf{y}')$, with $\mathbf{x}'' = \mathbf{x}' - (\mathbf{x}' \cdot \mathbf{y}')\mathbf{y}'$, or we can just keep them. This will make the algebra slightly more complicated because $\mathbf{x}'.\mathbf{y}' \neq 0$, but that's all. If it now happens that three vectors of the 2D plane, $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ are equally relevant, so that there is no good reason to eliminate one, then we must find a good way to mix them in order to produce two vectors out of three, which can serve as a basis for the 2D plane.

In the case of bosons, the space dimension is of course infinite, as well as the number of "unnecessary" states, so that the proper way to make such a space reduction cannot be an easy task. On that respect, to face the overcompleteness of the composite-boson states and to handle it up to the end as we do, seems to us a very secure way to control all the difficulties linked to the boson composite nature.

If we only consider the problem of nonorthogonality, we can think to overcome it by considering a physically relevant N-composite-boson state, for example $|0\rangle = B_0^{\dagger N}|v\rangle$, with all the bosons in the same state (this state is close to the N-composite-boson ground state). We can then replace the other composite-boson states, for example $|I\rangle = B^\dagger_i B_0^{\dagger N-1}|v\rangle$, by their component perpendicular to $|0\rangle$, namely $|I'\rangle = P_\perp B^\dagger_i B_0^{\dagger N-1}|v\rangle$, where

$$P_\perp = 1 - \frac{|0\rangle\langle 0|}{\langle 0|0\rangle}\ . \tag{3.1}$$

This helps partly only, because, even if we now have $\langle 0|I'\rangle = 0$, these $|I'\rangle$ states are not really good in the sense that they do not form an orthogonal set: we still have $\langle J'|I'\rangle \neq 0$. This remaining nonorthogonality can be unimportant in problems in which the $\langle J'|I'\rangle$ scalar products do not appear — for example, if they correspond to "higher order terms". However, even in these cases, such a construction of an orthogonal set is not fully satisfactory, when compared to handling the nonorthogonality, properly.

3.2 "Good" effective scattering

Our study of the interactions between two composite bosons makes appearing four scatterings: ξ_{mnij}, $\xi^{\rm in}_{mnij}$, $\xi^{\rm out}_{mnij}$ and λ_{mnij}. Let us, for a while, accept the idea to have bosonized particles which form an orthogonal set, so that the pure Pauli scatterings do not play a role, i.e., we drop all the λ_{mnij}'s. We are left with three scatterings having the dimension of an energy. An idea for a "good" effective scattering between elementary bosons can be to impose the same Hamiltonian matrix elements within the two-boson subspace. However, in view of equations (1.8) and (2.19), we are in trouble if we keep dropping the λ_{mnij}'s, because we can choose either $\xi_{mnij}-\xi^{\rm in}_{mnij}$ or $\xi_{mnij}-\xi^{\rm out}_{mnij}$, these two quantities being equal for $E_m+E_n=E_i+E_j$ only, due to equation (2.21). If, instead, we decide to keep the λ_{mnij}'s for a while, we are led to take

$$\hat{\xi}^{\rm eff}_{mnij}=\xi_{mnij}-\left[\xi^{\rm in}_{mnij}+(E_i+E_j)\lambda_{mnij}\right] , \qquad (3.2)$$

with the bracket possibly replaced by $\left[\xi^{\rm out}_{mnij}+(E_m+E_n)\lambda_{mnij}\right]$; so that we can rewrite this effective scattering, in a more symmetrical form, as

$$\hat{\xi}^{\rm eff}_{mnij}= \xi_{mnij}-\frac{1}{2}\left[\xi^{\rm in}_{mnij}+\xi^{\rm out}_{mnij}+(E_m+E_n+E_i+E_j)\lambda_{mnij}\right] . \qquad (3.3)$$

We note that this $\hat{\xi}^{\rm eff}_{mnij}$ is such that $\hat{\xi}^{\rm eff}_{mnij}=\left(\hat{\xi}^{\rm eff}_{ijmn}\right)^*$, as necessary for the hermiticity of the effective Hamiltonian for elementary bosons. If we now drop the Pauli scatterings λ_{mnij}'s, we are led to take

$$\xi^{\rm eff}_{mnij}=\xi_{mnij}-(\xi^{\rm in}_{mnij}+\xi^{\rm out}_{mnij})/2 , \qquad (3.4)$$

which preserves the hermiticity of the Hamiltonian. This has to be contrasted with the effective scattering for bosonized excitons extensively used by the semiconductor community [11,12], namely $\xi_{mnij}-\xi^{\rm out}_{mnij}$, as first obtained by Hanamura and Haug, following an Inui's bosonization procedure [20].

Before going further, let us note that, in dropping the λ_{mnij} term in $\hat{\xi}^{\rm eff}$ to get $\xi^{\rm eff}$, we actually "drop" a quite unpleasant feature of this effective scattering: its spurious dependence on the band gap in the case of excitons. Indeed, in $\hat{\xi}^{\rm eff}_{mnij}$ appears the sum — not the difference — of the "in" and "out" boson energies. In the case of excitons, this boson energy is essentially equal to the band gap plus a small term depending on the exciton state at hand. So that $E_m+E_n+E_i+E_j$ is essentially equal to four times the band gap. Its possible appearance in a scattering is a physical nonsense.

All this leads us to conclude that the only "reasonable" scattering between two elementary bosons — which has the dimension of an energy, preserves hermiticity and has no spurious band gap dependence — should be $\xi^{\rm eff}_{mnij}$.

Actually, even this $\xi^{\rm eff}_{mnij}$ is not good, except may be for effects in which only enter *first* order *diagonal* Coulomb processes — in order for the "in" and "out" Coulomb scatterings to be equal. Indeed, in a previous work [10], we have shown that the link between the inverse lifetime of an exciton state — due to exciton-exciton interations — and the sum of its scattering rates towards a different exciton state, misses a factor of 2, if the excitons are replaced by elementary bosons, *whatever* is the effective scattering used — a quite strong statement! We have recently recovered this result [21] without calculating the two quantities explicitly, but just by using an argument based on differences in the closure relations of elementary and composite excitons.

Let us now come back to the problem of having the Pauli scatterings systematically missing in any approach which uses an effective bosonic Hamiltonian. This is actually far worse than the problem of choosing a "good" exchange part for Coulomb scattering, because, with this dropping, we not only miss a factor of 2, but the dominant term [15,16] in all optical nonlinear effects! Indeed, a photon interacts with a semiconductor through the virtual exciton to which this photon is coupled. If the semiconductor already has excitons, the first way this virtual exciton interacts is via Pauli exclusion, since this exclusion among fermions makes it filling all the fermion states already occupied in the sample. Coulomb interaction comes next, since it has to come with an energy denominator which, in problems involving unabsorbed photons, is a detuning, so that these Coulomb terms always give a negligible contribution at large detuning, in front of the terms coming from Pauli scatterings alone.

Beside the exciton optical Stark effect, in which the roots of our many-body theory for composite excitons can be found [22], we have studied some other optical nonlinearities in which the interaction of a composite exciton with the matter is dominated by Pauli scattering, namely the theory of the third order nonlinear susceptibility $\chi^{(3)}$ [16], the theory of Faraday rotation in photoexcited semiconductors [23] and the precession of a spin pined on an impurity induced by unabsorbed photons [24].

Since this Pauli scattering, quite crucial in many physical effects, is dimensionless, it cannot appear in the effective Hamiltonian of bosonized particles, which needs a scattering having the dimension of an energy. Consequently, all terms in which this scattering appears alone, i.e., not mixed with Coulomb processes, are going to be missed by any procedure using an effective bosonic Hamiltonian. (Note that this is also true for any approach using a spin-spin Hamiltonian [13]).

Finally, our qualitative discussion on the possible interactions between three composite bosons, has led us to identify, in addition to pure exchange processes based on Pauli scattering between three composite bosons, again missed, more complicated mixtures of Coulomb and exchange than the one appearing between two composite bosons, $\xi^{\rm in}_{mnij}$ and $\xi^{\rm out}_{mnij}$. In order not to miss them, we could think of adding a three-body part to the Hamiltonian like

$$\bar{V}' = \frac{1}{3!} \sum_{mnpijk} \bar{\xi}^{\rm eff}_{mnpijk} \, \bar{B}^\dagger_m \bar{B}^\dagger_n \bar{B}^\dagger_p \bar{B}_i \bar{B}_j \bar{B}_k \ . \qquad (3.5)$$

Let us however note that the proper identification of $\bar{\xi}^{\rm eff}_{mnpijk}$ with the three-body processes not constructed from ξ_{mnij}, $\xi^{\rm in}_{mnij}$ and $\xi^{\rm out}_{mnij}$, is quite tricky, because this three-body potential formally contains terms in which one elementary boson can stay unchanged, i.e., terms already included in $\bar{V}$.

All this actually means that the "good" effective bosonic Hamiltonian, apart from the pure Pauli terms which are going to be missed anyway, has to be more and more complicated if we want to include processes in which more and more bosons are involved, i.e., if we want to study many-body effects, really. Just for that, the replacement of composite bosons by elementary boson seems to us far more complicated than keeping the boson composite nature through a set of Pauli scatterings, as we propose.

4 Extension to more complicated composite bosons

In the preceding sections, we have considered composite bosons made of a pair of different fermions, these pairs being eigenstates of the Hamiltonian. In this last section, we are going to show how we can generalize the definitions of the various scatterings we have found, to pairs of fermions which are not eigenstates of the Hamiltonian. For clarity, we are going to show this generalization on a specific example of current interest: a composite boson made of a pair of trapped electrons [13,14].

Let us consider two electrons with two traps located at $\mathbf{R}_1$ and $\mathbf{R}_2$. These traps can be semiconductor quantum dots, Coulomb traps such as ionized impurities, H atom protons, and so on... The system Hamiltonian then reads

$$H = H_0 + V_{ee} + W_{\mathbf{R}_1} + W_{\mathbf{R}_2} \ , \qquad (4.1)$$

where H_0 is the kinetic contribution, V_{ee} the electron-electron Coulomb interaction and $W_{\mathbf{R}}$ the potential of the trap located at $\mathbf{R}$. The physically relevant one-electron states [24] are the one-electron eigenstates in the presence of one trap located at $\mathbf{R}$, namely $|\mathbf{R}\mu\rangle$ given by $(H_0 + W_{\mathbf{R}} - \epsilon_\mu)|\mathbf{R}\mu\rangle = 0$. They are such that

$$|\mathbf{R}\mu\rangle = a^\dagger_{\mathbf{R}\mu}|v\rangle = \sum_{\mathbf{k}} \langle \mathbf{k}|\mathbf{R}\mu\rangle \, a^\dagger_{\mathbf{k}}|v\rangle \ , \qquad (4.2)$$

$a^\dagger_{\mathbf{k}}$ being the creation operator for a free electron with momentum $\mathbf{k}$. In the case of Coulomb trap, the $|\mathbf{R}\mu\rangle$ states are just the H atom bound and extended states.

We now consider the two-electron states having one electron on each trap,

$$|n\rangle = A^\dagger_n|v\rangle = a^\dagger_{\mathbf{R}_1\mu_1} \, a^\dagger_{\mathbf{R}_2\mu_2}|v\rangle \ , \qquad (4.3)$$

where the index n stands for (μ_1, μ_2). These states do not form an orthogonal set since, due to the finite overlap of the one-electron wave functions, we do have

$$\langle n'|n\rangle = \delta_{n',n} - \lambda^{(e-e)}_{n'n} \ , \qquad (4.4)$$

where $\lambda^{(e-e)}_{n'n} = \langle \mathbf{R}_1\mu'_1|\mathbf{R}_2\mu_2\rangle \, \langle \mathbf{R}_2\mu'_2|\mathbf{R}_1\mu_1\rangle$. This possible carrier exchange between the two traps, shown in Figure 7a, produces not only the nonorthogonality of the $|n\rangle$ states, but also the overcompleteness of this set of states. Indeed, by exchanging the electrons between the traps $\mathbf{R}_1$ and $\mathbf{R}_2$, we can show that

$$A^\dagger_n = -\sum_{n'} \lambda^{(e-e)}_{n'n} \, A^\dagger_{n'} \ . \qquad (4.5)$$

If we now want to determine the Pauli scatterings of this composite boson made of a pair of trapped electrons, we are led to define the deviation-from-boson operator $D_{n'n}$ through

$$D_{n'n} = \langle n'|n\rangle - [A_{n'}, A^\dagger_n] \ , \qquad (4.6)$$

which is a generalization of equation (2.3) to the case of nonorthogonal composite bosons. Indeed, with such a definition, we still have the crucial property of a deviation-from-boson operator, namely $D_{n'n}|v\rangle = 0$. The Pauli scatterings of the composite boson $A^\dagger_n$ with another composite boson $B^\dagger_i$ is then obtained through

$$[D_{n'n}, B^\dagger_i] = 2\sum_{i'} \lambda^{(ee-X)}_{n'i'ni} \, B^\dagger_{i'} \ . \qquad (4.7)$$

In a case of current interest, namely the spin manipulation by a laser pulse [13,14,25,26], the relevant bosons $B^\dagger_i$ with which the pair of trapped electrons interact, are the virtual excitons coupled to the photons. This composite boson $B^\dagger_i$, made of an electron-hole pair can exchange its electron with one of the two electrons of the composite boson $A^\dagger_n$, through the Pauli scattering $\lambda^{(ee-X)}_{n'i'ni}$ shown in Figure 7b. This is why we have defined it with a 2 prefactor in equation (4.7).

We now look for the scatterings of the composite bosons $A^\dagger_n$ having the dimension of an energy. For that, we first note that $H_0 + W_{\mathbf{R}}$ can be written in terms of the $a^\dagger_{\mathbf{R}\mu}$'s defined in equation (4.2), as [24]

$$H_0 + W_{\mathbf{R}} = \sum_\mu \epsilon_\mu \, a^\dagger_{\mathbf{R}\mu} a_{\mathbf{R}\mu} \ , \qquad (4.8)$$

where ϵ_μ is the energy of the one-electron state $|\mathbf{R}\mu\rangle$. This leads to

$$H|n\rangle = E_n|n\rangle + \sum_{n'} \xi^{(e-e)}_{n'n}|n'\rangle \ , \qquad (4.9)$$

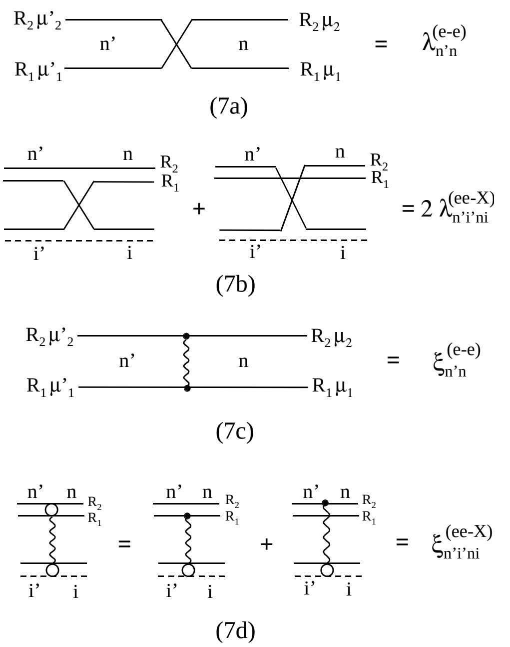

Fig. 7. (a) Pauli scattering $\lambda_{n'n}^{(e-e)}$ between two electrons trapped in $\mathbf{R}_1$ and $\mathbf{R}_2$. In this exchange, the electrons can end in trapped states $n' = (\mu_1', \mu_2')$ different from the initial ones $n = (\mu_1, \mu_2)$. (b) Pauli scattering $\lambda_{n'i'ni}^{(ee-X)}$ between a composite boson made of a trapped electron pair and a composite boson made of an electron-hole pair, i.e., more precisely a composite exciton, their states changing from (n, i) to (n', i'). (c) Direct scattering $\xi_{n'n}^{(e-e)}$ between two trapped electrons. This scattering contains the Coulomb interaction between the two electrons as well as the interactions of each electron with the potential of the other trap. (d) Direct scattering $\xi_{n'i'ni}^{(ee-X)}$ between a composite boson made of a trapped electron pair and a composite exciton. This scattering contains the Coulomb interaction of the exciton with each of the two trapped electrons.

where $E_n = \epsilon_{\mu_1} + \epsilon_{\mu_2}$ is the "free" energy of the pair of trapped electrons, while $\xi_{n'n}^{(e-e)}$, shown in Figure 7c, comes from their Coulomb repulsion as well as from the interaction of each electron with the other trap.

For such composite bosons $A_n^\dagger$, which, due to equation (4.9), are not eigenstates of the Hamiltonian, the proper way to define their "creation potential" is through

$$V_n^\dagger = [H, A_n^\dagger] - E_n A_n^\dagger - \sum_{n'} \xi_{n'n}^{(e-e)} A_{n'}^\dagger \ , \qquad (4.10)$$

in order to still have $V_n^\dagger |v\rangle = 0$, equation (4.10) being a generalization of equation (2.15). We then get the "direct Coulomb scattering" between the composite boson $A_n^\dagger$ and another composite boson $B_i^\dagger$, through

$$[V_n^\dagger, B_i^\dagger] = \sum_{n'i'} \xi_{n'i'ni}^{(ee-X)} \, A_{n'}^\dagger B_{i'}^\dagger \ , \qquad (4.11)$$

which is similar to equation (2.17). This direct scattering is shown in Figure 7d. It corresponds to the direct Coulomb interaction of each of the two trapped electrons with the electron-hole pair of the exciton.

Using this set of commutators and the two scatterings $\lambda_{n'n}^{(e-e)}$ and $\xi_{n'n}^{(e-e)}$ they generate, we are going to calculate the energy of two trapped electrons with their possible exchanges included exactly, in order to determine the singlet-triplet splitting these exchange processes induce in the van der Waals energy. Using them and the two scatterings between the trapped pair and an exciton, $\lambda_{n'n}^{(ee-X)}$ and $\xi_{n'n}^{(ee-X)}$, we are also going to calculate the splitting of the trapped electron pair energy induced by virtual excitons coupled to a laser beam, which results from electron exchanges between the trapped pair and the electron of the virtual exciton. This last problem is of great current interest for the control of the spin transfer time between two traps using a laser pulse, with, in mind, its possible use for quantum information [27].

5 Conclusion

In this paper, we have made a detailed qualitative analysis of what can be called "interaction" between two or three composite bosons. We have shown that all the processes which produce a change in the boson states, can be written in terms of two scatterings only: a direct Coulomb scattering which has the dimension of an energy and a pure Pauli "scattering" which is dimensionless. This Pauli scattering is actually the novel ingredient of our many-body theory for composite bosons in which these composite bosons are never replaced by elementary bosons.

We can possibly think of including processes in which enter complicated mixtures of direct Coulomb scatterings and Pauli scatterings, through a set of effective scatterings between two, three, or more elementary bosons. On the opposite, all processes in which the Pauli scatterings appear alone have to be missed if one uses effective Hamiltonians such as the ones in which the composite bosons are replaced by elementary bosons, or any spin-spin Hamiltonian, whatever the effective coupling is. This, in particular, happens in all semiconductor optical nonlinearities, the virtual exciton coupled to the photon field feeling the presence of the fermions present in the sample, "even more" than their charges.

Finally, we have shown how to extend the mathematical definitions of the Pauli scattering and the direct Coulomb scattering to non trivial composite bosons which are not eigenstates of the Hamiltonian, such as a pair of trapped electrons. This extension again goes through the introduction of "deviation-from-boson operators" and "creation potentials", the main characteristic of these quantities being to give zero when they act on vacuum, so that they really describe interactions with the rest of the system.

Although it is easy to understand the reluctance one may have to enter a new way of thinking interactions between composite bosons, it appears to us as worthwhile to spend the necessary amount of time to grasp these new ideas, in view of their potentiality in very many problems of physics.

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